# Chapter s – Approximations of Special Functions

## 1. Scope of the Chapter

This chapter provides functions for computing some of the most commonly required special functions of mathematics. See Chapter g01 for probability distribution functions and their inverses.

## 2. Accuracy

The majority of the functions in this chapter evaluate real-valued functions of a single real variable $x$. The accuracy of the computed value is discussed in each function document, along the following lines.

Let $\Delta$ be the *absolute* error in the argument $x$, and $\delta$ be the *relative* error in $x$.

If we ignore errors that arise in the argument by propagation of data errors etc. and consider only those errors that result from the fact that a real number is being represented in the computer in floating-point form with finite precision, then $\delta$ is bounded and this bound is independent of the magnitude of $x$; for example, on an $n$-digit machine

$$|\delta| \leq 10^{-n}.$$

(This of course implies that the absolute error $\Delta = x\delta$ is also bounded but the bound is now dependent on $x$.)

Let $E$ be the absolute error in the computed function value $f(x)$, and $\epsilon$ be the relative error. Then

$$E \simeq |f'(x)|\Delta$$
$$E \simeq |xf'(x)|\delta$$
$$\epsilon \simeq |xf'(x)/f(x)|\delta.$$

If possible, the function documents discuss the last of these relations, that is the propagation of relative error, in terms of the error amplification factor $|xf'(x)/f(x)|$. But in some cases, such as near zeros of the function which cannot be extracted explicitly, absolute error in the result is the quantity of significance and here the factor $|xf'(x)|$ is described.

In general, testing of the functions has shown that the behaviour of the actual errors follows fairly well these theoretical relations. In regions where the error amplification factors are less than one, or of the order of one, the errors are slightly larger than the above predictions. The errors are here limited largely by the finite precision of arithmetic in the machine, but $\epsilon$ is normally no more than a few times greater than the bound on $\delta$. In regions where the amplification factors are large, of order of ten or greater, the theoretical analysis gives a good measure of the accuracy obtainable.

## 3. Approximations to Elliptic Integrals

### 3.1. Definitions of Symmetrised Elliptic Integrals

Four functions in the chapter compute symmetrised variants of the classical elliptic integrals. These alternative definitions have been suggested by Carlson and he also developed the basic algorithms used in this chapter.

**Important Advice**: users who encounter elliptic integrals in the course of their work are strongly recommended to look at transforming their analysis directly to one of the Carlson forms, rather than the traditional canonical Legendre forms. In general, the extra symmetry of the Carlson forms is likely to simplify the analysis, and these symmetric forms are much more stable to calculate. In case that is not possible, rules for computing the Legendre forms in terms of the Carlson forms are given below.

The symmetrised integral of the first kind, which is computed by nag_elliptic_integral_rf (s21bbc), is defined by

$$R_F(x,y,z) = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

where $x, y, z \geq 0$ and at most one may be equal to zero. The normalisation factor, $\frac{1}{2}$, is chosen so as to make

$$R_F(x,x,x) = 1/\sqrt{x}.$$

If any two of the variables are equal, $R_F$ degenerates into the function $R_C$, which is computed by nag_elliptic_integral_rc (s21bac):

$$R_C(x,y) = R_F(x,y,y) = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{t+x}(t+y)}$$

where the argument restrictions are now $x \geq 0$ and $y \neq 0$.

The symmetrised integral of the second kind, which is computed by nag_elliptic_integral_rd (s21bcc), is defined by

$$R_D(x,y,z) = \tfrac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}$$

with $z > 0$, $x \geq 0$ and $y \geq 0$ but only one of $x$ or $y$ may be zero.

This function is a degenerate special case of the integral of the third kind, which is computed by nag_elliptic_integral_rj (s21bdc) and is defined by

$$R_J(x,y,z,\rho) = \tfrac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+\rho)}$$

with $\rho \neq 0$, $x, y, z \geq 0$ with at most one equality holding. Thus $R_D(x,y,z) = R_J(x,y,z,z)$. The normalisation of both these functions is chosen so that

$$R_D(x,x,x) = R_J(x,x,x,x) = 1/(x\sqrt{x}).$$

### 3.2. Relationships with Classical Elliptic Integrals

The above forms can be related to the more traditional Legendre canonical forms as follows. Let

$$q = \cos^2\phi, \qquad r = 1 - m\sin^2\phi, \qquad s = 1 + n\sin^2\phi,$$

where $0 < \phi \leq \frac{1}{2}\pi$. Then we have

the incomplete elliptic integral of the first kind:

$$F(\phi|m) = \int_0^{\sin\phi} (1-t^2)^{-1/2}(1-mt^2)^{-1/2}dt = \sin\phi R_F(q,r,1);$$

the incomplete elliptic integral of the second kind:

$$\begin{aligned} E(\phi|m) &= \int_0^{\sin\phi} (1-t^2)^{-1/2}(1-mt^2)^{1/2}dt \\ &= \sin\phi R_F(q,r,1) - \tfrac{1}{3}m\sin^3\phi R_D(q,r,1); \end{aligned}$$

the elliptic integral of the third kind:

$$\begin{aligned} \Pi(n;\phi|m) &= \int_0^{\sin\phi} (1-t^2)^{-1/2}(1-mt^2)^{-1/2}(1+nt^2)^{-1}dt \\ &= \sin\phi R_F(q,r,1) - \tfrac{1}{3}n\sin^3\phi R_J(q,r,1,s). \end{aligned}$$

Also the complete elliptic integral of the first kind:

$$K(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}d\theta = R_F(0, 1-m, 1);$$

and the complete elliptic integral of the second kind:

$$\begin{aligned} E(m) &= \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2} d\theta \\ &= R_F(0, 1-m, 1) - \tfrac{1}{3} m R_D(0, 1-m, 1). \end{aligned}$$

The function $R_C$ is related to the logarithm or inverse hyperbolic functions if $0 < y < x$, and to the inverse circular functions if $0 \leq x \leq y$. For example

$$\begin{aligned} \ln x &= (x-1) R_C\left(\left(\frac{1+x}{2}\right)^2, x\right), \quad x > 0; \\ \arcsin x &= x R_C(1-x^2, 1), \quad |x| \leq 1; \\ \operatorname{arcsinh} x &= x R_C(1+x^2, 1), \quad \text{etc.} \end{aligned}$$

In general this method of calculating elementary functions is not recommended as there are usually much more efficient specific functions available. However $R_C$ may be used, for example, to compute $\ln x/(x-1)$ when $x$ is close to 1, without the loss of significant figures that occurs when $\ln x$ and $x-1$ are computed separately.

## 4. Available Functions

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|---|---|
| Airy function, $\mathrm{Ai}(x)$ | `s17agc` |
| Airy function, $\mathrm{Ai}'(x)$ | `s17ajc` |
| Airy function, $\mathrm{Bi}(x)$ | `s17ahc` |
| Airy function, $\mathrm{Bi}'(x)$ | `s17akc` |
| $\mathrm{Arccosh}(x)$ | `s11acc` |
| $\mathrm{Arcsinh}(x)$ | `s11abc` |
| $\mathrm{Arctanh}(x)$ | `s11aac` |
| Bessel function, $J_0(x)$ | `s17aec` |
| Bessel function, $J_1(x)$ | `s17afc` |
| Bessel function, $Y_0(x)$ | `s17acc` |
| Bessel function, $Y_1(x)$ | `s17adc` |
| Complementary error function, $\mathrm{erfc}(x)$ | `s15adc` |
| $\mathrm{Cosh}(x)$ | `s10acc` |
| Cosine Integral, $\mathrm{Ci}(x)$ | `s13acc` |
| Cumulative Normal distribution | `s15abc` |
| Cumulative Normal distribution complement | `s15acc` |
| Elliptic integral, symmetrised, degenerate of 1st kind, $R_C$ | `s21bac` |
| Elliptic integral, symmetrised, of 1st kind, $R_F$ | `s21bbc` |
| Elliptic integral, symmetrised, of 2nd kind, $R_D$ | `s21bcc` |
| Elliptic integral, symmetrised, of 3rd kind, $R_J$ | `s21bdc` |
| Error function, $\mathrm{erf}(x)$ | `s15aec` |
| Exponential Integral, $E_1(x)$ | `s13aac` |
| Fresnel Integral, $C(x)$ | `s20adc` |
| Fresnel Integral, $S(x)$ | `s20acc` |
| Gamma function, $\Gamma(x)$ | `s14aac` |
| Incomplete Gamma function, $P(a,x)$ and $Q(a,x)$ | `s14bac` |
| Kelvin function, $\mathrm{bei}\, x$ | `s19abc` |
| Kelvin function, $\mathrm{ber}\, x$ | `s19aac` |
| Kelvin function, $\mathrm{kei}\, x$ | `s19adc` |
| Kelvin function, $\mathrm{ker}\, x$ | `s19acc` |
| Logarithm of Gamma function, $\ln\Gamma(x)$ | `s14abc` |
| Modified Bessel function, $I_0(x)$ | `s18aec` |
| Modified Bessel function, $I_1(x)$ | `s18afc` |
| Modified Bessel function, $K_0(x)$ | `s18acc` |
| Modified Bessel function, $K_1(x)$ | `s18adc` |
| Scaled modified Bessel function, $e^{-\lvert x\rvert} I_0(x)$ | `s18cec` |

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|---|---|
| Scaled modified Bessel function, $e^{-|x|}I_1(x)$ | s18cfc |
| Scaled modified Bessel function, $e^x K_0(x)$ | s18ccc |
| Scaled modified Bessel function, $e^x K_1(x)$ | s18cdc |
| Sine integral, $Si(x)$ | s13adc |
| Sinh$(x)$ | s10abc |
| Tanh$(x)$ | s10aac |

See also Chapter g01 for probability distribution functions and their inverses.